Mat G2 14 hét gyakorlat

1) Adjuk meg az alábbi függvények Fourier-sorát!

a) $f(x) = \sin^2 x$ b) $f(x) = \sin^3 x$ c) $f(x) = \sin 2x + \cos^2 x + 1$

Mo: a) $f(x) = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$, ahol $a_0 = 1$, $a_1 = 0$, $a_2 = -\frac{1}{2}$

$a_n = 0 \ \forall n \geq 3$ és $b_n \equiv 0$. $\forall n \in \mathbb{N}$

b) $f(x) = \sin^3 x$. Mivel $\sin 3x = \sin(2x + x) = \sin 2x \cos x + \sin x \cos 2x =$

$= 2\sin x \cos^2 x + \sin x(\cos^2 x - \sin^2 x) = 3\sin x \cos^2 x - \sin^3 x =$

$= 3\sin x - 4\sin^3 x$, így $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$\Rightarrow a_n \equiv 0 \ \forall n \in \mathbb{N}$, $b_1 = \frac{3}{4}$, $b_2 = 0$ és $b_3 = -\frac{1}{4}$, $b_n = 0 \ \forall n \geq 4$-re

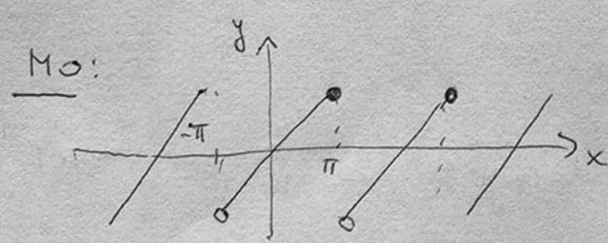
c) $f(x) = \sin 2x + \cos^2 x + 1 = \sin 2x + \frac{1+\cos 2x}{2} + 1 = \sin 2x + \frac{1}{2}\cos 2x + \frac{3}{2}$

$\Rightarrow a_0 = 3$, $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_n \equiv 0 \ \forall n \geq 3$ és $b_1 = 0$, $b_2 = 1$, $b_n \equiv 0 \ \forall n \geq 3$

2) Adjuk meg az alábbi függvények Fourier-sorát!

$f(x) = x \qquad x \in (-\pi, \pi] \qquad f(x) = f(x + 2\pi)$

Mo:



f páratlan $\Rightarrow a_n \equiv 0 \ \forall n \in \mathbb{N}$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{\text{páros fv.}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx =$$

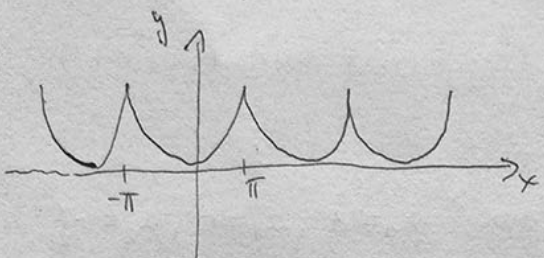
$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{x}\,\underset{v'}{\sin nx}\,dx = \frac{2}{\pi}\left(\left[\underset{u}{x}\left(\underset{v}{-\frac{\cos nx}{n}}\right)\right]_0^{\pi} - \underbrace{\int_0^{\pi} \underset{u'}{-}\underset{v}{\frac{\cos nx}{n}}\,dx}_{=0}\right) = \frac{2}{n\pi}\left(\pi(-1)^{n+1}\right)$$

$$\Rightarrow b_n = \frac{2}{n}(-1)^{n+1} \qquad f(x) = \sum_1^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

3, Adjuk meg az előbbi függvény Fourier-sorát!

$$f(x) = x^2 \qquad x \in (-\pi, \pi] \qquad f(x) = f(x+2\pi)$$

Mo:



f páros fv. $\Rightarrow$ $b_n \equiv 0 \quad \forall n \in \mathbb{N}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx =$$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u}{x^2}\, \underset{v'}{\cos nx}\, dx = \frac{2}{\pi}\left(\left[\underset{u}{x^2}\, \underset{v}{\frac{\sin nx}{n}}\right]_0^{\pi} - \int_0^{\pi} \underset{u'}{2x}\, \underset{v}{\frac{\sin nx}{n}}\, dx\right) =$$

$$= \left(-\frac{4}{n\pi}\int_0^{\pi} \underset{u}{x}\, \underset{v'}{\sin nx}\, dx\right) = -\frac{4}{n\pi}\left[\left[\underset{u}{x}\left(\underset{v}{-\frac{\cos nx}{n}}\right)\right]_0^{\pi} - \underbrace{\int_0^{\pi} \underset{u'v}{-\frac{\cos nx}{n}}\, dx}_{=0}\right] =$$

$$= -\frac{4}{n\pi} \cdot \frac{\pi(-1)^{n+1}}{n} = \frac{4(-1)^n}{n^2} \quad \text{, azaz } a_n = \frac{4(-1)^n}{n^2}$$

Így $\quad f(x) = \left(\sum_{n=1} \frac{4(-1)^n}{n^2}\cos nx\right) + \frac{\pi^2}{3} \quad$ ui,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} x^2\, dx = \frac{2}{\pi}\,\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

azaz
$$f(x) = \frac{\pi^2}{3} + \sum_1^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

4, Határozzuk meg a fenti eredmény alapján a $\sum_1^{\infty} \frac{1}{n^2}$ sor összegét!

Mo $\quad x = \pi$ esetén $\quad f(x) = \pi^2$ így

$$\pi^2 = \frac{\pi^2}{3} + \sum_1^{\infty} \frac{4(-1)^n \cdot \cos n\pi}{n^2} = \frac{\pi^2}{3} + \sum_1^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2 - \frac{\pi^2}{3}}{4} = \frac{\pi^2}{6}$$

5, Határozzuk meg a fenti eredmény alapján a $\sum_1^{\infty} \frac{(-1)^n}{n^2}$ sor összegét!

Mo: $\quad x = 0$ esetén $\quad f(x) = 0$, így

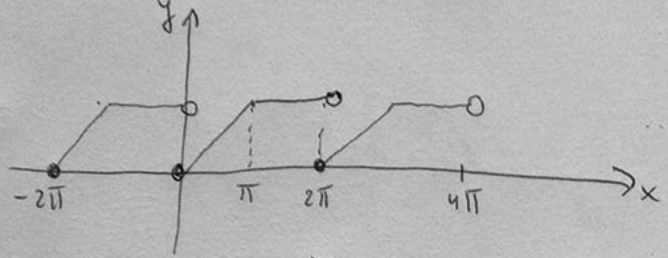
$$0 = \frac{\pi^2}{3} + \sum_1^{\infty} \frac{4(-1)^n}{n^2}\cos n \cdot 0 = \frac{\pi^2}{3} + \sum_1^{\infty} \frac{4(-1)^n}{n^2}$$

$$\Rightarrow \sum_1^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

6, Határozzuk meg az alábbi függvény Fourier-sorát!

$$f(x) = \begin{cases} x & \text{ha } x \in [0, \pi) \\ \pi & x \in [\pi, 2\pi) \end{cases} \qquad f(x) = f(x+2\pi)$$

Mo:



A függvény nem páros, nem páratlan

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} \pi\,dx\right) = \frac{1}{\pi}\left(\frac{\pi^2}{2} + \pi^2\right) = \frac{3\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(\int_0^{\pi} \underset{u}{x}\,\underset{v'}{\cos nx}\,dx + \int_{\pi}^{2\pi} \pi\cos nx\,dx\right) =$$

$$= \frac{1}{\pi}\left(\underbrace{\left[\underset{u}{x}\,\underset{v}{\frac{\sin nx}{n}}\right]_0^{\pi}}_{=0} - \int_0^{\pi} \underset{u'}{}\underset{v}{\frac{\sin nx}{n}}\,dx + \underbrace{\left[\pi\frac{\sin nx}{n}\right]_{\pi}^{2\pi}}_{=0}\right) = -\frac{1}{n\pi}\int_0^{\pi} \sin nx\,dx =$$

$$= +\frac{1}{n\pi}\left[+\frac{\cos nx}{n}\right]_0^{\pi} = \frac{1}{n^2\pi}\left((-1)^n - 1\right) \quad \text{és}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left(\int_0^{\pi} \underset{u}{x}\,\underset{v'}{\sin nx}\,dx + \int_{\pi}^{2\pi} \pi\sin nx\,dx\right) =$$

$$= \frac{1}{\pi}\left(\left[\frac{\underset{u}{x}\,\underset{v}{(-\cos nx)}}{n}\right]_0^{\pi} - \underbrace{\int_0^{\pi} \underset{u'}{}\underset{v}{\left(-\frac{\cos nx}{n}\right)}\,dx}_{=0} + \pi\left[-\frac{\cos nx}{n}\right]_{\pi}^{2\pi}\right) =$$

$$= \frac{1}{\pi}\left(\frac{\pi\cdot(-1)^{n+1}}{n} + \frac{\pi}{n}\left((-1) - (-1)^{n+1}\right)\right) = -\frac{1}{n}$$

Így $$f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi}\left((-1)^n - 1\right)\cos nx - \frac{1}{n}\sin nx\right).$$